Refresher Course in Calculus, Probability, and Statistics

Day 5b: Matrix Operations

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- **Matrix:** a rectangular array (collection) of numbers.
- An $m \times n$ (read: m-by-n) matrix has m rows and n columns:

$$\underset{m \times n}{\mathbf{A}} = (a_{ij})_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where $(a_{ij})_{m \times n}$ denotes the element a in row i and column j.

If:

- $m = 1$ and $n > 1$: **row vector**;
- $m > 1$ and $n = 1$: **column vector**;
- $m = n$: **square matrix**;
- $m = n = 1$: **scalar**.

James Joseph Sylvester
1814–1897

Matrix Operations: equality of matrices

- Let $\mathbf{A}_{m\times n} = (a_{ij})_{m\times n}, \quad \mathbf{B}_{p\times q} = (b_{ij})_{p\times q}, \quad \alpha_{1\times 1}$.
- $\mathbf{A}$ and $\mathbf{B}$ are **equal** iff:
  - same order (same dimensions): $m = p$ and $n = q$;
  - identical elements in corresponding locations: $a_{ij} = b_{ij}$ for all i, j.
- Examples:

$$\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} \neq \begin{pmatrix} 5 & 0 \\ 4 & 2 \end{pmatrix} \neq \begin{pmatrix} 4 & 2 & 1 \\ 5 & 0 & -5 \end{pmatrix}.$$

Matrix Operations: addition and subtraction

- Matrices can be added or subtracted only if they have the **same order** (conformability condition for addition).
- **Addition** — sum each pair of corresponding elements: $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.

$$\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 4+1 & 2+2 \\ 5+2 & 0+(-3) \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 7 & -3 \end{pmatrix}.$$

- **Subtraction** is similar: $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$.

$$\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}.$$

Matrix Operations: scalar multiplication

- Multiplication by a scalar α: multiply each element by α, $\alpha \mathbf{A}_{m\times n} = (\alpha\, a_{ij})_{m\times n}$.

$\rightarrow$ One can invert scalar multiplication by extracting a common factor.

- Examples:

$$3\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ 15 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -2 & -1 \\ -\frac{5}{2} & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix}.$$

Matrix Operations: multiplication

- $\mathbf{A}$ and $\mathbf{B}$ can be multiplied only if the column dimension of $\mathbf{A}$ equals the row dimension of $\mathbf{B}$ (conformability for multiplication):
  - with $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{p\times q}$, $\mathbf{AB}$ is defined iff $n = p$;
  - the product $\mathbf{C} = \mathbf{AB}$ has as many rows as $\mathbf{A}$ and as many columns as $\mathbf{B}$: $\mathbf{A}_{m\times n}\mathbf{B}_{p\times q} = \mathbf{C}_{m\times q}$.
- c_{ij} is built from the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$: pair the elements, multiply each pair, and sum the products:

$$c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

- *Note:* there is no matrix division.

Matrix Operations: multiplication (continued)

Example 1: $\mathbf{A}_{2\times 3}\,\mathbf{B}_{3\times 2} = \mathbf{C}_{2\times 2}$

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 2 + 3\cdot 6 & 1\cdot 3 + 0\cdot 5 + 3\cdot 2 \\ 2\cdot 1 + 1\cdot 2 + 5\cdot 6 & 2\cdot 3 + 1\cdot 5 + 5\cdot 2 \end{pmatrix} = \begin{pmatrix} 19 & 9 \\ 34 & 21 \end{pmatrix}$$

Example 2: $\mathbf{A}_{3\times 2}\,\mathbf{B}_{2\times 1} = \mathbf{C}_{3\times 1}$

$$\begin{pmatrix} 1 & 3 \\ -2 & 8 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 3\cdot 1 \\ -2\cdot 2 + 8\cdot 1 \\ 0\cdot 2 + 12\cdot 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 12 \end{pmatrix}$$

Matrix Operations: properties

- **Matrix addition**
  - commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$;
  - associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.
- **Matrix multiplication**
  - *not* commutative: $\mathbf{AB} \neq \mathbf{BA}$ (with exceptions);
  - associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$;
  - distributive (pre-multiplication): $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$;
  - distributive (post-multiplication): $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$.
- **Idiosyncrasies of matrix algebra**
  - there is no matrix division;
  - $\mathbf{AB} = \mathbf{0}$ does *not* imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ (can occur when $\mathbf{A}, \mathbf{B}$ are singular);
  - $\mathbf{AB} = \mathbf{AC}$ with $\mathbf{A} \neq \mathbf{0}$ does *not* imply $\mathbf{B} = \mathbf{C}$ (can occur when the matrices are singular, see below).

Matrix Operations: powers

- For a square matrix $\mathbf{A}_{n\times n}$:

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A}, \quad \mathbf{A}^3 = \mathbf{A}\mathbf{A}\mathbf{A}, \ \ldots, \ \mathbf{A}^N = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{N \text{ times}}.$$

- Example: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}.$
- **Idempotent matrix** ($\mathbf{A}^2 = \mathbf{A}$): $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}.$
- **Diagonal matrix** — powers act element-wise: if $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_n)$ then $\mathbf{D}^m = \mathrm{diag}(d_1^m, \ldots, d_n^m)$; e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$

Matrix Operations: the identity matrix

- **Identity matrix:** a square matrix with ones on its main diagonal and zeros elsewhere:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

- Same role as 1 in scalar algebra:
  - $\mathbf{A}_{m\times n}\mathbf{I}_n = \mathbf{I}_m\mathbf{A}_{m\times n} = \mathbf{A}_{m\times n}$;
  - you can insert or delete an identity without changing a product: $\mathbf{A}_{m\times n}\mathbf{I}_{n\times n}\mathbf{B}_{n\times p} = \mathbf{A}_{m\times n}\mathbf{B}_{n\times p}$;
  - $\mathbf{I}_n^k = \mathbf{I}_n$ (idempotent).

- **Null (zero) matrix:** a matrix with zeros everywhere; same role as 0 in scalar algebra:

$$\mathbf{0}_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{0}_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- Properties:
  - $\mathbf{A}_{m\times n} + \mathbf{0}_{m\times n} = \mathbf{0}_{m\times n} + \mathbf{A}_{m\times n} = \mathbf{A}_{m\times n}$;
  - $\mathbf{A}_{m\times n}\,\mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}$ and $\mathbf{0}_{q\times m}\,\mathbf{A}_{m\times n} = \mathbf{0}_{q\times n}$.

Matrix Operations: transpose

- Interchanging the rows and columns of a matrix $\mathbf{A}$ yields its **transpose**, denoted $\mathbf{A}'$ or $\mathbf{A}^T$:

$$\underset{2\times 3}{\mathbf{A}} = \begin{pmatrix} 3 & 8 & -2 \\ 0 & 1 & 9 \end{pmatrix} \implies \underset{3\times 2}{\mathbf{A}'} = \begin{pmatrix} 3 & 0 \\ 8 & 1 \\ -2 & 9 \end{pmatrix}$$

  (reflection of the elements along the main diagonal).

- Properties:
  - $(\mathbf{A}')' = \mathbf{A}$ (transpose is an involution);
  - $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$ (transpose respects addition);
  - $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$ (transpose of a product is the product of transposes in *reverse* order);
  - $(\alpha\mathbf{A})' = \alpha\mathbf{A}'$ (transpose of a scalar is the same scalar).

- If $\mathbf{A}'_{n\times n} = \mathbf{A}_{n\times n}$, the matrix is **symmetric**:

$$\mathbf{D} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -4 & 3 \\ -1 & 3 & 9 \end{pmatrix} = \mathbf{D}'.$$

Matrix Operations: determinant

- The **determinant** of a square matrix $\mathbf{A}$, denoted $|\mathbf{A}|$ or $\det \mathbf{A}$, is a scalar derived from $\mathbf{A}$.
- 2×2 matrix:

$$\mathbf{A}_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad |\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}.$$

$\rightarrow$ the difference between the product of the diagonal elements and the product of the off-diagonal elements (a determinant of order 2).

- Example:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \qquad |\mathbf{A}| = 3 \cdot 1 - 2 \cdot 1 = 1.$$

Matrix Operations: inverse

- The **inverse** of a square matrix $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, is defined by $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$\rightarrow$ as in scalar algebra, where $\alpha\alpha^{-1} = 1$ if $\alpha \neq 0$).

- A matrix $\mathbf{A}$ is **invertible** iff:
  - it is square $(n \times n)$ — *necessary condition*;
  - it is nonsingular, i.e. $|\mathbf{A}| \neq 0$ — *sufficient condition*.
- Properties of inverses:
  - $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ (involution);
  - $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$;
  - $(\alpha\mathbf{A})^{-1} = \alpha^{-1}\mathbf{A}^{-1}$ for $\alpha \neq 0$.

Case of a 2×2 matrix

2×2 matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};$$

Determinant:

$$|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}.$$

Inverse matrix:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Extra slides – Computation of inverse matrix

Computing the inverse matrix: the adjugate (cofactor) method

- The definition of the inverse matrix involves 2 main objects:
  - the determinant $|\mathbf{A}|$
  - the adjucate $\mathrm{adj}(\mathbf{A})$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathrm{adj}(\mathbf{A})$$

Determinant — Laplace (cofactor) expansion

Each row-1 entry (pivot) $\times$ its 2×2 minor (delete that entry's row & column), with alternating signs $+ - +$.

$$\begin{vmatrix} \boxed{a} & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

entry $\mathbf{a}$, minor $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$

$$\begin{aligned} |\mathbf{A}| &= \mathbf{a}\,(ei - fh) \\ &\quad - b\,(di - fg) \\ &\quad + c\,(dh - eg) \end{aligned}$$

- minor M_{ij}: delete row i, column j.
- cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.
- $|\mathbf{A}| = \sum_j a_{ij} C_{ij}$ (expand along any row i).

Determinant — Laplace (cofactor) expansion

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entry **b**, minor $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$

$$\begin{aligned} |\mathbf{A}| &= a\,(ei - fh) \\ &\quad - \mathbf{b}\,(di - fg) \\ &\quad + c\,(dh - eg) \end{aligned}$$

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entry $\mathbf{c}$, minor $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$\begin{aligned} |\mathbf{A}| &= a\,(ei - fh) \\ &\quad - b\,(di - fg) \\ &\quad + \mathbf{c}\,(dh - eg) \end{aligned}$$

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- $|\mathbf{A}| = \sum_j a_{ij} C_{ij}$ (expand along any row i).

The recipe — four steps

1. Compute the determinant $|\mathbf{A}|$.

$\boxed{1}$ det $\rightarrow$ $\boxed{2}$ cofactors $\rightarrow$ $\boxed{3}$ transpose $\rightarrow$ $\boxed{4}$ $\div$ det

The recipe — four steps

1. Compute the determinant $|\mathbf{A}|$.
2. Replace every entry by its cofactor $C_{ij} = (-1)^{i+j} M_{ij} \;\Rightarrow\;$ cofactor matrix $\mathbf{C}^{+}$.

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The recipe — four steps

1. Compute the determinant $|\mathbf{A}|$.
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3. Transpose it: $\mathrm{adj}(\mathbf{A}) = (\mathbf{C}^{+})^{T}$.

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1. Compute the determinant $|\mathbf{A}|$.
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3. Transpose it: $\operatorname{adj}(\mathbf{A}) = (\mathbf{C}^{+})^{T}$.
4. Divide by the determinant: $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|}\operatorname{adj}(\mathbf{A})$.

$\boxed{1}$ det $\rightarrow$ $\boxed{2}$ cofactors $\rightarrow$ $\boxed{3}$ transpose $\rightarrow$ $\boxed{4}$ $\div$ det

Worked example — Step 1: the determinant

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

$$\begin{vmatrix} \boxed{1} & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= 1\,(9 - 16) \\ &\quad - 3\,(3 - 4) \\ &\quad + 3\,(4 - 3) \end{aligned}$$

1 det → 2 cofactors → 3 transpose → 4 ÷ det

Worked example — Step 1: the determinant

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & \boxed{3} & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= 1\,(9-16) \\ &\quad - \mathbf{3}\,(3-4) \\ &\quad + 3\,(4-3) \end{aligned}$$

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Worked example — Step 1: the determinant

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Worked example — Step 1: the determinant

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 3 & \boxed{3} \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= 1\,(9-16) \\ &- 3\,(3-4) \\ &+ 3\,(4-3) \\ &= -7+3+3 \;=\; \mathbf{-1} \end{aligned}$$

1 det → 2 cofactors → 3 transpose → 4 ÷ det

Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{11} = + \left| \begin{smallmatrix} 3 & 4 \\ 4 & 3 \end{smallmatrix} \right| = 9 - 16 = -7$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & & \\ & & \\ & & \end{pmatrix}$$

1 det $\rightarrow$ 2 cofactors $\rightarrow$ 3 transpose $\rightarrow$ 4 $\div$ det

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$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{12} = -\left|\begin{smallmatrix} 1 & 4 \\ 1 & 3 \end{smallmatrix}\right| = -(3-4) = 1$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & \\ & & \\ & & \end{pmatrix}$$

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Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & \boxed{3} \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{13} = +\left|\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right| = 4 - 3 = 1$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ & & \\ & & \end{pmatrix}$$

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Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{21} = -\left|\begin{smallmatrix} 3 & 3 \\ 4 & 3 \end{smallmatrix}\right| = -(9-12) = 3$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ 3 & & \\ & & \end{pmatrix}$$

1 det → 2 cofactors → 3 transpose → 4 ÷ det

Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & \boxed{3} & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{22} = + \left| \begin{smallmatrix} 1 & 3 \\ 1 & 3 \end{smallmatrix} \right| = 3 - 3 = 0$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ 3 & 0 & \\ & & \end{pmatrix}$$

1 det → 2 cofactors → 3 transpose → 4 ÷ det

Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{23} = -\left|\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right| = -(4-3) = -1$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ & & \end{pmatrix}$$

1 det $\rightarrow$ **2 cofactors** $\rightarrow$ 3 transpose $\rightarrow$ 4 $\div$ det

Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{31} = + \left| \begin{smallmatrix} 3 & 3 \\ 3 & 4 \end{smallmatrix} \right| = 12 - 9 = 3$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ 3 & & \end{pmatrix}$$

1 det $\rightarrow$ **2 cofactors** $\rightarrow$ 3 transpose $\rightarrow$ 4 $\div$ det

Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

$$C_{32} = -\left|\begin{smallmatrix} 1 & 3 \\ 1 & 4 \end{smallmatrix}\right| = -(4-3) = -1$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ 3 & -1 & \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

1 det → **2** cofactors → 3 transpose → 4 ÷ det

Worked example — Step 2: cofactors $C_{ij} = (-1)^{i+j} M_{ij}$

Delete row i (grey) and column j; the surviving 2×2 block is the minor; the checkerboard gives the sign.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

sign pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_{33} = + \left| \begin{smallmatrix} 1 & 3 \\ 1 & 3 \end{smallmatrix} \right| = 3 - 3 = 0$$

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}$$

1 det $\rightarrow$ **2** cofactors $\rightarrow$ 3 transpose $\rightarrow$ 4 $\div$ det

Worked example — Steps 3–4: transpose, then divide

Reflect across the main diagonal (amber fixed; each off-diagonal pair swaps), then divide by $|\mathbf{A}| = -1$.

$$\mathbf{C}^{+} = \begin{pmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix} \xrightarrow{(\,\cdot\,)^T} \operatorname{adj}(\mathbf{A}) = \begin{pmatrix} -7 & 3 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

1 det $\rightarrow$ 2 cofactors $\rightarrow$ **3** transpose $\rightarrow$ 4 $\div$ det

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$$\mathbf{A}^{-1} = \frac{1}{-1} \begin{pmatrix} -7 & 3 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

$\boxed{1}$ det $\rightarrow$ $\boxed{2}$ cofactors $\rightarrow$ $\boxed{3}$ transpose $\rightarrow$ $\boxed{4}$ $\div$ det

Check: $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$

Column k of $\mathbf{A}^{-1}$ dotted against the rows of $\mathbf{A}$ returns the k-th column of the identity: diagonal entries give 1, the rest give 0.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & & \\ 0 & & \\ 0 & & \end{pmatrix} = \mathbf{I}$$

$$\text{row}_1 \cdot \text{col}_1 = 1(7) + 3(-1) + 3(-1) = 1, \qquad \text{row}_1 \cdot \text{col}_2 = 1(-3) + 3(0) + 3(1) = 0$$

Check: $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$

Column k of $\mathbf{A}^{-1}$ dotted against the rows of $\mathbf{A}$ returns the k-th column of the identity: diagonal entries give 1, the rest give 0.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & \\ 0 & \mathbf{1} & \\ 0 & 0 & \end{pmatrix} = \mathbf{I}$$

$$\text{row}_2 \cdot \text{col}_2 = 1(-3) + 3(0) + 4(1) = 1$$

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All nine dot products check out: $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$. ✓