Math Camp — Day 5 Exercises

Worked Solutions

The matrices are

$$A = \begin{pmatrix} 7 & 2 & 6 \\ 5 & 4 & 8 \\ 3 & 1 & 9 \end{pmatrix}_{3\times3}, \quad B = \begin{pmatrix} 6 & 2 \\ 5 & 0 \end{pmatrix}_{2\times2}, \quad C = \begin{pmatrix} 11 \\ 4 \\ 13 \end{pmatrix}_{3\times1},$$

$$D = \begin{pmatrix} 14 \\ 2 \end{pmatrix}_{2\times1}, \quad E = \begin{pmatrix} 8 & 1 & 10 \end{pmatrix}_{1\times3}, \quad F = \begin{pmatrix} 13 & 3 \end{pmatrix}_{1\times2}.$$

Exercise 1 — Conformability and products

A product $M_{m\times n}N_{p\times q}$ is defined iff $n = p$; the result is $m \times q$.

a) AC**:** (3×3)(3×1) — **defined,** 3×1**.**

$$AC = \begin{pmatrix} 7(11) + 2(4) + 6(13) \\ 5(11) + 4(4) + 8(13) \\ 3(11) + 1(4) + 9(13) \end{pmatrix} = \begin{pmatrix} 163 \\ 175 \\ 154 \end{pmatrix}.$$

b) BD**:** (2×2)(2×1) — **defined,** 2×1**.**

$$BD = \begin{pmatrix} 6(14) + 2(2) \\ 5(14) + 0(2) \end{pmatrix} = \begin{pmatrix} 88 \\ 70 \end{pmatrix}.$$

c) EC**:** (1×3)(3×1) — **defined,** 1×1**.**

$$EC = \big(8(11) + 1(4) + 10(13)\big) = (222).$$

d) DF**:** (2×1)(1×2) — **defined,** 2×2**.**

$$DF = \begin{pmatrix} 14 \\ 2 \end{pmatrix} \begin{pmatrix} 13 & 3 \end{pmatrix} = \begin{pmatrix} 182 & 42 \\ 26 & 6 \end{pmatrix}.$$

e) CA**:** (3×1)(3×3) — **inner dims** $1 \neq 3$**: not defined.**

f) DE**:** (2×1)(1×3) — **defined,** 2×3**.**

$$DE = \begin{pmatrix} 14 \\ 2 \end{pmatrix} \begin{pmatrix} 8 & 1 & 10 \end{pmatrix} = \begin{pmatrix} 112 & 14 & 140 \\ 16 & 2 & 20 \end{pmatrix}.$$

g) DB**:** (2×1)(2×2) — **inner dims** $1 \neq 2$**: not defined.**

h) CF**:** $(3{\times}1)(1{\times}2)$ **— defined,** 3×2**.**

$$CF = \begin{pmatrix} 11 \\ 4 \\ 13 \end{pmatrix} \begin{pmatrix} 13 & 3 \end{pmatrix} = \begin{pmatrix} 143 & 33 \\ 52 & 12 \\ 169 & 39 \end{pmatrix}.$$

i) EF**:** $(1{\times}3)(1{\times}2)$ **— inner dims** $3 \neq 1$**: not defined.**

Exercise 2 — Inverses

For a 2×2 matrix, $M^{-1} = \dfrac{1}{\det M}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

$A = \left(\begin{smallmatrix} 24 & 15 \\ 8 & 7 \end{smallmatrix}\right)$**.** $\det A = 24\cdot 7 - 15\cdot 8 = 168 - 120 = 48$, so

$$A^{-1} = \frac{1}{48}\begin{pmatrix} 7 & -15 \\ -8 & 24 \end{pmatrix} = \begin{pmatrix} \frac{7}{48} & -\frac{5}{16} \\ -\frac{1}{6} & \frac{1}{2} \end{pmatrix}.$$

$B = \left(\begin{smallmatrix} 7 & 9 \\ 6 & 12 \end{smallmatrix}\right)$**.** $\det B = 7\cdot 12 - 9\cdot 6 = 84 - 54 = 30$, so

$$B^{-1} = \frac{1}{30}\begin{pmatrix} 12 & -9 \\ -6 & 7 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{3}{10} \\ -\frac{1}{5} & \frac{7}{30} \end{pmatrix}.$$

Exercise 3 — Determinants

Cofactor expansion along the first row.

$C = \left(\begin{smallmatrix} 0 & -1 & -1 \\ -1 & 8 & 3 \\ -1 & 3 & 12 \end{smallmatrix}\right)$**.**

$$\det C = 0\begin{vmatrix} 8 & 3 \\ 3 & 12 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 3 \\ -1 & 12 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 8 \\ -1 & 3 \end{vmatrix}.$$

$$\begin{vmatrix} -1 & 3 \\ -1 & 12 \end{vmatrix} = -12 + 3 = -9, \qquad \begin{vmatrix} -1 & 8 \\ -1 & 3 \end{vmatrix} = -3 + 8 = 5,$$

$$\det C = 0 + (1)(-9) + (-1)(5) = -9 - 5 = \mathbf{-14}.$$

$D = \left(\begin{smallmatrix} 0 & -4 & -2 \\ -4 & 0 & 1 \\ -2 & 1 & 0 \end{smallmatrix}\right)$**.**

$$\det D = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - (-4)\begin{vmatrix} -4 & 1 \\ -2 & 0 \end{vmatrix} + (-2)\begin{vmatrix} -4 & 0 \\ -2 & 1 \end{vmatrix}.$$

$$\begin{vmatrix} -4 & 1 \\ -2 & 0 \end{vmatrix} = 0 + 2 = 2, \qquad \begin{vmatrix} -4 & 0 \\ -2 & 1 \end{vmatrix} = -4 - 0 = -4,$$

$$\det D = 0 + (4)(2) + (-2)(-4) = 8 + 8 = \mathbf{16}.$$