# Math Camp — Day 3 Exercises
Worked Solutions

## Exercise 1 — Number of heads in two coin tosses

Sample space $\{HH, HT, TH, TT\}$, each with probability $\frac{1}{4}$; $Y \in \{0, 1, 2\}$.

**a) Probability distribution (pdf).**

$$P(Y{=}0) = \tfrac{1}{4}, \qquad P(Y{=}1) = \tfrac{2}{4} = \tfrac{1}{2}, \qquad P(Y{=}2) = \tfrac{1}{4}.$$

or

$$f(0) = \frac{1}{4}, \qquad f(1) = \frac{1}{2}, \qquad f(2) = \frac{1}{4}.$$

**b) Cumulative distribution (cdf).**

$$P(Y{\leq}0) = \tfrac{1}{4}, \qquad P(Y{\leq}1) = \tfrac{3}{4}, \qquad P(Y{\leq}2) = 1.$$

or

$$F(0) = \frac{1}{4}, \qquad F(1) = \frac{3}{4}, \qquad F(2) = 1.$$

**c) Mean and variance.**

$$E(Y) = 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} = 1.$$

$$E(Y^2) = 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} + 4 \cdot \tfrac{1}{4} = \tfrac{3}{2}, \qquad \mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2}.$$

## Exercise 2 — Employment and college graduation

Joint distribution:

| | $Y$=0 (unempl.) | $Y$=1 (empl.) | Total |
|---|---|---|---|
| $X$=0 (non-grad) | 0.045 | 0.709 | 0.754 |
| $X$=1 (grad) | 0.005 | 0.241 | 0.246 |
| Total | 0.05 | 0.95 | 1.00 |

**a) $E(Y)$.**

$$E(Y) = 0 \cdot P(Y{=}0) + 1 \cdot P(Y{=}1) = 0.95.$$

**b) Unemployment rate = $1 - E(Y)$.** Because $Y$ is binary, $E(Y) = P(Y{=}1) =$ the employment rate. The unemployment rate is the complementary fraction:

$$P(Y{=}0) = 1 - P(Y{=}1) = 1 - E(Y) = 0.05.$$

**c) Conditional means.**

$$E(Y \mid X{=}1) = P(Y{=}1 \mid X{=}1) = \frac{0.241}{0.246} = 0.9797,$$

$$E(Y \mid X{=}0) = P(Y{=}1 \mid X{=}0) = \frac{0.709}{0.754} = 0.9403.$$

**d) Unemployment rate by group ($= 1 - E(Y \mid X)$).**

(i) College grads: $1 - 0.9797 = 0.0203 \approx 2.0\%$,

(ii) Non-college grads: $1 - 0.9403 = 0.0597 \approx 6.0\%$.

**e) $P$(education | unemployed), i.e. condition on $Y{=}0$.**

$$\text{(i) } P(X{=}1 \mid Y{=}0) = \frac{0.005}{0.05} = 0.10, \qquad \text{(ii) } P(X{=}0 \mid Y{=}0) = \frac{0.045}{0.05} = 0.90.$$

**f) Independence?** Compare a joint cell with the product of its marginals:

$$P(X{=}0, Y{=}0) = 0.045 \quad \text{vs.} \quad P(X{=}0)P(Y{=}0) = 0.754 \times 0.05 = 0.0377.$$

They differ, so education and employment status are **not independent**.

## Exercise 3 — Storm damage

$P(Y{=}0) = 0.95$, $P(Y{=}\$20{,}000) = 0.05$.

**a) Mean and standard deviation.**

$$E(Y) = 0.95(0) + 0.05(20000) = \$1000.$$

$$E(Y^2) = 0.05(20000)^2 = 20{,}000{,}000, \qquad \mathrm{Var}(Y) = 20{,}000{,}000 - 1000^2 = 19{,}000{,}000.$$

$$\sigma_Y = \sqrt{19{,}000{,}000} \approx \$4358.9.$$

**b) Pool of $n = 100$ independent homes, average damage $\bar{Y}$.** *1)* $E(\bar{Y}) = E(Y) = \$1000$.

*2)* By independence, $\mathrm{Var}(\bar{Y}) = \dfrac{\mathrm{Var}(Y)}{100} = 190{,}000$, so $\sigma_{\bar{Y}} = \sqrt{190{,}000} \approx 435.9$. By the CLT $\bar{Y} \overset{\text{approx}}{\sim} N(1000, 190000)$:

$$z = \frac{2000 - 1000}{435.9} \approx 2.29.$$

From the standard normal table $\Phi(2.29) = 0.9890$, hence

$$P(\bar{Y} > 2000) = 1 - \Phi(2.29) \approx 0.011 \quad (\approx 1.1\%).$$