Math Camp — Day 2 Exercises

Worked Solutions

Exercise 1 — Extrema

For each function: (1) set $f'(x) = 0$ to get critical values; (2) use f'' (and higher derivatives when $f'' = 0$) to classify as a maximum, a minimum, or an inflection point.

a) $f(x) = 3x^3 - 36x^2 + 135x - 13$**.**

- $f'(x) = 9x^2 - 72x + 135 = x^2 - 8x + 15 = 0.$

  Trick: A polynomial of order 2 can be rewritten as:

  $$(x-a)(x-b) = x^2 - bx - ax + ab = x^2 - (a+b)x + ab.$$

  Let's identify a and b:

  $$x = \begin{cases} a+b = 8 \\ ab = 15 \end{cases}$$

  $\Rightarrow a = 5, b = 3$. The 2 critical values are: $x = 3,\ x = 5$.

- $f''(x) = 18x - 72.$

  $$\begin{aligned} f''(3) &= -18 < 0 && \Rightarrow \textbf{maximum at } x = 3\ (f(3) = 149), \\ f''(5) &= 18 > 0 && \Rightarrow \textbf{minimum at } x = 5\ (f(5) = 137). \end{aligned}$$

b) $f(x) = 2x^4 - 16x^3 + 32x^2 + 5$**.** (Applying the same trick as in a):

- $f'(x) = 8x^3 - 48x^2 + 64x = 8x(x-2)(x-4) \Rightarrow$ There are 3 critical values: $x = 0,\ 2,\ 4$.
- $f''(x) = 24x^2 - 96x + 64$:

  $$\begin{aligned} f''(0) &= 64 > 0 && \Rightarrow \textbf{minimum at } x = 0\ (f(0) = 5), \\ f''(2) &= -32 < 0 && \Rightarrow \textbf{maximum at } x = 2\ (f(2) = 37), \\ f''(4) &= 64 > 0 && \Rightarrow \textbf{minimum at } x = 4\ (f(4) = 5). \end{aligned}$$

c) $f(x) = -(x-8)^4$**.**

$$f'(x) = -4(x-8)^3 = 0 \ \Rightarrow\ x = 8, \qquad f''(x) = -12(x-8)^2,\ f''(8) = 0.$$

The second-derivative test is inconclusive. The first non-vanishing derivative at $x = 8$ is $f^{(4)}(8) = -24$. It is of *even* order and *negative*, so $x = 8$ is a **maximum** ($f(8) = 0$).

d) $f(x) = (5-x)^3$**.**

$$f'(x) = -3(5-x)^2 = 0 \ \Rightarrow\ x = 5, \qquad f''(x) = 6(5-x),\ f''(5) = 0.$$

Inconclusive; the first non-zero derivative is $f'''(5) = -6$, of *odd* order:
$x = 5$ is an **inflection point** ($f(5) = 0$).

Exercise 2 — Optimisation of bivariate functions

1. Set both first partials to zero to find the critical points
2. Compute both second partials and the cross derivatives to determine whether the function is maximized, minimized, is at an inflection point, or is at a saddle point.

a) $f = 3x_1^2 - x_1x_2 + 2x_2^2 - 4x_1 - 7x_2 + 12$**.**

- $f_1 = 6x_1 - x_2 - 4 = 0, \qquad f_2 = -x_1 + 4x_2 - 7 = 0.$
  From the first, $x_2 = 6x_1 - 4$; substituting: $-x_1 + 4(6x_1 - 4) - 7 = 0 \Rightarrow 23x_1 = 23$.

$$x_1 = 1 \quad \Rightarrow \quad x_2 = 6 - 4 = 2 \quad \Rightarrow \quad (x_1, x_2) = (1, 2).$$

- $f_{11} = 6,\ f_{22} = 4,\ f_{12} = -1.$

$$f_{11} \times f_{22} = 24 > (-1)^2 = f_{12}^2 \quad \text{and } f_{11} > 0 \ \&\ f_{22} > 0 \quad \Rightarrow \quad (x_1, x_2) \text{ is a } \textbf{minimum}.$$

b) $f = 60x_1 + 34x_2 - 4x_1x_2 - 6x_1^2 - 3x_2^2 + 5$**.**

- $f_1 = 60 - 4x_2 - 12x_1 = 0, \qquad f_2 = 34 - 4x_1 - 6x_2 = 0.$

  First equation: $x_2 = 15 - 3x_1$; substitute: $34 - 4x_1 - 6(15 - 3x_1) = 0 \Rightarrow 14x_1 = 56$.

$$x_1 = 4 \quad \Rightarrow \quad x_2 = 15 - 12 = 3 \quad \Rightarrow \quad (x_1, x_2) = (4, 3).$$

- $f_{11} = -12,\ f_{22} = -6,\ f_{12} = -4.$

$$f_{11} \times f_{22} = 72 > 16 = (f_{12})^2 \quad \text{and } f_{11} < 0 \ \&\ f_{22} < 0 \quad \Rightarrow \quad (x_1, x_2) \text{ is a } \textbf{maximum}.$$

Exercise 3 — Lagrange multipliers

a) Optimise $f(x_1, x_2) = 4x_1^2 - 2x_1x_2 + 6x_2^2$ subject to $x_1 + x_2 = 72$. Form the Lagrangian

$$\mathcal{L} = 4x_1^2 - 2x_1x_2 + 6x_2^2 + \lambda(72 - x_1 - x_2).$$

First-order conditions:

$$\mathcal{L}_{x_1} = 8x_1 - 2x_2 - \lambda = 0, \quad \mathcal{L}_{x_2} = -2x_1 + 12x_2 - \lambda = 0, \quad \mathcal{L}_\lambda = 72 - x_1 - x_2 = 0.$$

Equating the first two eliminates λ:

$$8x_1 - 2x_2 = -2x_1 + 12x_2 \ \Rightarrow\ 10x_1 = 14x_2 \ \Rightarrow\ x_1 = \tfrac{7}{5}x_2.$$

Insert into the constraint: $\frac{7}{5}x_2 + x_2 = 72 \Rightarrow 2.4\,x_2 = 72$, hence

$$x_2 = 30, \qquad x_1 = 42, \qquad \lambda = 8(42) - 2(30) = 276.$$

Optimal value: $f(42, 30) = 4(1764) - 2(1260) + 6(900) = \mathbf{9936}$. (The objective is a positive-definite quadratic, so on the constraint line this stationary point is in fact a minimum.)

b) λ is the marginal value of the constraint constant, so a one-unit increase in the right-hand side changes the optimal objective value by approximately

$$\Delta f^* \approx \lambda = \mathbf{276}.$$

c) Re-optimise with $x_1 + x_2 = 73$. The ratio $x_1 = \frac{7}{5}x_2$ is unchanged, so $2.4\,x_2 = 73 \Rightarrow x_2 = \frac{365}{12} \approx 30.417$, $x_1 = \frac{511}{12} \approx 42.583$.

$$f_{\text{new}} = \frac{1470804}{144} = 10213.92.$$

Actual change $= 10213.92 - 9936 = \mathbf{277.92}$, very close to the Lagrange estimate 276. The small gap arises because λ is a derivative (marginal) estimate, exact only for an infinitesimal change.