Refresher Course in Calculus, Probability, and Statistics

Day 2: Optimization

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- **Optimization**: the process of finding the relative maximum or minimum of a function.
- Optimization problems are among the most frequent in economics and finance:
  - maximize profit or minimize cost (producer)
  - maximize utility (consumer)
  - maximize return or minimize variance (investor)
  - minimize the sum of squared errors when estimating a regression
  - ...
- In economics and finance, optimization problems often imply constraints

$\Rightarrow$ **Constrained optimization**.

Optimization

- Requires primarily a (relevant) **objective function**
  - functional relationship between the object of optimization and its determinants

$$\underbrace{y}_{\text{object of optimization}} = \underbrace{f}_{\text{objective function}} (\overset{\substack{\text{choice variable(s)} \\ \downarrow}}{x})$$

  - usually assumed to be (twice) continuously differentiable
- Example: maximize profit π
  - profit is the difference between total revenue R and total cost C
  - both revenue and cost are functions of the quantity produced Q: $R = R(Q), \quad C = C(Q)$
  - objective function of the profit-maximization problem:

$$\pi(Q) = R(Q) - C(Q)$$

  - the only choice variable is the quantity Q

First-order condition (FOC)

To find local/global **extrema** of a function, take the first derivative and set it equal to zero:

$$f'(x) = 0$$

- Necessary condition, known as the **first-order condition (FOC)**.
  Necessary but *not* sufficient.
- Identifies all points at which the function is neither increasing nor decreasing, but at a plateau.
- All points identified this way are **candidates** (or critical points) for a possible maximum or minimum.

Unconstrained Optimization (continued)

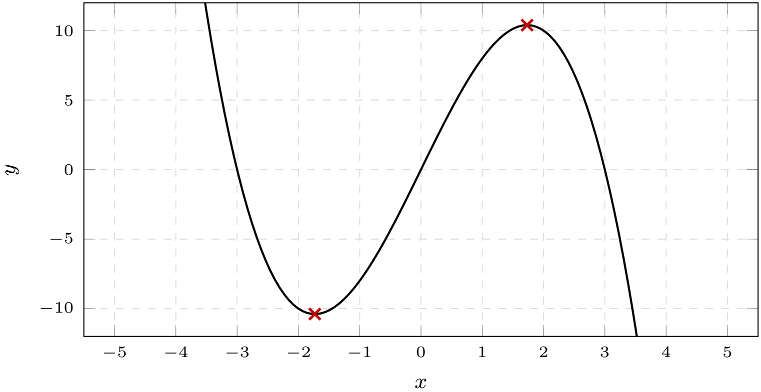
Second-order condition

- Once critical points are identified by the FOC (i.e. $f'(x) = 0$ for some $x = a$):
  - take the second derivative of $f(x)$
  - evaluate it at each critical point
- Assuming the necessary FOC is met, the **second-order condition (SOC)** is a sufficient condition to qualify an extremum:

If...	**then the function is...**	**and a is a...**
$f''(a) < 0$	concave at a	relative maximum
$f''(a) > 0$	convex at a	relative minimum
$f''(a) = 0$	test is inconclusive	

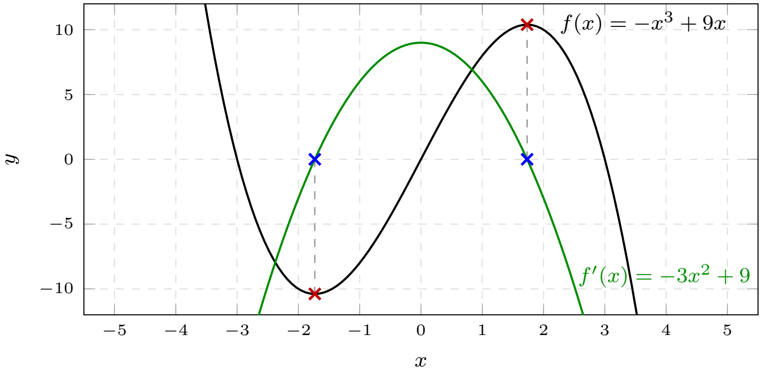
Unconstrained Optimization (continued)

Example: find the extreme points, maximum(s) and minimum(s) of $f(x) = -x^3 + 9x$



Unconstrained Optimization (continued)

- FOC: $f'(x) = 0 = -3x^2 + 9$
  - roots: $x_{1,2} = \pm\sqrt{3}$
  - extreme points: $(x_1^*, y_1^*) = (\sqrt{3},\ 6\sqrt{3})$ and $(x_2^*, y_2^*) = (-\sqrt{3},\ -6\sqrt{3})$



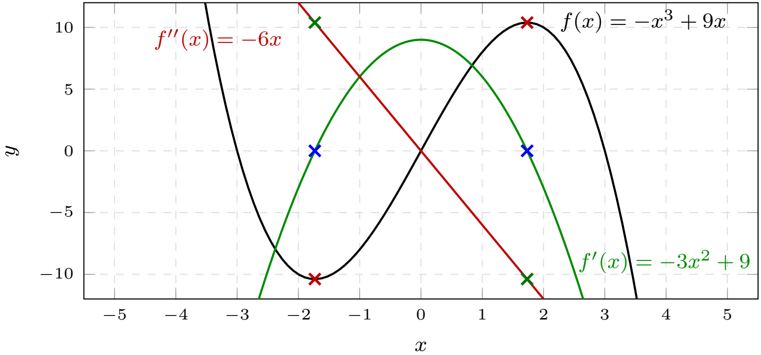
NB: $f(\sqrt{3}) = -3\sqrt{3} + 9\sqrt{3} = 6\sqrt{3}$; $\quad f(-\sqrt{3}) = 3\sqrt{3} - 9\sqrt{3} = -6\sqrt{3}$.

Unconstrained Optimization (continued)

- SOC: $f''(x) = -6x$
  - at $x = x_1^* = \sqrt{3}$: $f''(\sqrt{3}) = -6\sqrt{3} < 0$

  $\Rightarrow$ $(x_1^*, y_1^*) = (\sqrt{3}, 6\sqrt{3})$ is a **maximum**.
  - at $x = x_2^* = -\sqrt{3}$: $f''(-\sqrt{3}) = 6\sqrt{3} > 0$

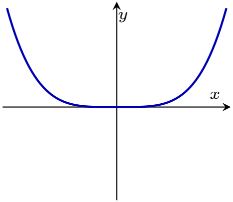
  $\Rightarrow$ $(x_2^*, y_2^*) = (-\sqrt{3}, -6\sqrt{3})$ is a **minimum**.



Unconstrained Optimization (continued)

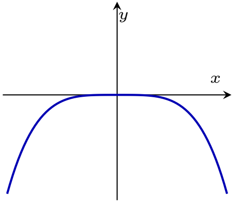
Possible cases where $f'(a) = f''(a) = 0$ (here with $a = 0$)

$y = x^4$



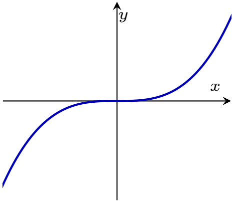
Minimum

$y = -x^4$



Maximum

$y = x^3$



Inflection point

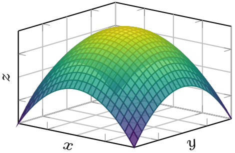
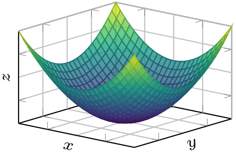
Unconstrained Optimization (continued)

Successive-derivative test

- If $f''(a) = 0$, the SOC is inconclusive.
- The successive-derivative test is helpful: evaluate higher-order derivatives at the critical point.
- If:
  - the 1st non-zero higher-order derivative is an **odd**-numbered derivative (3rd, 5th, etc.):
    - $\rightarrow$ **inflection point**, i.e. a point at which the function changes from being convex ($f''(a) > 0$) to concave ($f''(a) < 0$), or vice versa
  - the 1st non-zero higher-order derivative is an **even**-numbered derivative (4th, 6th, etc.) and the value of this derivative is:
    - $\rightarrow$ negative $\Rightarrow$ **maximum**
    - $\rightarrow$ positive $\Rightarrow$ **minimum**

Multivariable Functions (continued)

- Searching for extrema of a multivariable function $z = f(x, y)$:

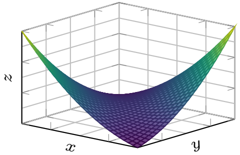
Condition(s)	**Maximum**	**Minimum**
First-order necessary	$\frac{\partial f(x,y)}{\partial x} = f_x = 0$ and	$\frac{\partial f(x,y)}{\partial y} = f_y = 0$
Second-order **necessary**	$\frac{\partial^2 f}{\partial x^2} = f_{xx} < 0$ and $\frac{\partial^2 f}{\partial y^2} = f_{yy} < 0$	$\frac{\partial^2 f}{\partial x^2} = f_{xx} > 0$ and $\frac{\partial^2 f}{\partial y^2} = f_{yy} > 0$
Second-order **sufficient**	$f_{xx} \times f_{yy} > (f_{xy})^2$	
Shape		

NB: the sufficient SOC follows from: $\det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx} f_{yy} - (f_{xy})^2 > 0.$

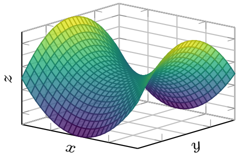
Multivariable Functions (continued)

- Searching for extrema of a multivariable function $z = f(x, y)$. If:

◦ $f_{xx} \times f_{yy} < (f_{xy})^2$ **and** f_{xx} & f_{yy} *same* signs
$\rightarrow$ **inflection point**
e.g. $z = x^2 + y^2 + 3xy$



◦ $f_{xx} \times f_{yy} < (f_{xy})^2$ **and** f_{xx} & f_{yy} *opposite* signs
$\rightarrow$ **saddle point**
e.g. $z = x^2 - y^2$



◦ $f_{xx} \times f_{yy} = (f_{xy})^2$: test is inconclusive
e.g. $z = x^3 + y^3$

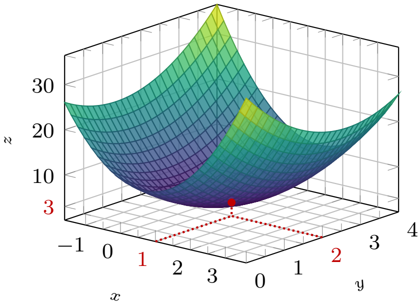
NB: 3D plots can be generated at `https://www.geogebra.org/3d?lang=fr`

Multivariable Functions (continued)

Example 1: Consider: $f(x,y) = 3x^2 - xy + 2y^2 - 4x - 7y + 12$

- FOCs:
  - $f_x = 6x - y - 4 = 0$
  - $f_y = -x + 4y - 7 = 0$
  - one critical point at $(x^*, y^*) = (1, 2)$

- SOC:
  - $f_{xx} = 6$
  - $f_{yy} = 4$
  - $f_{xy} = f_{yx} = -1$



$\rightarrow\ f_{xx} \times f_{yy} = 24 > (f_{xy})^2 = (-1)^2 = 1$

$\Rightarrow\ f(x,y)$ has an extremum at $(x^*, y^*) = (1, 2)$, which is a **minimum**.

Example 2: $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

Strategy: solve the system $f_x = 0,\ f_y = 0$ by factoring the equation: a product is zero exactly when one of its factors is, so the factored equation splits *every* solution into exhaustive cases.

- **FOCs:** $f_x = 6xy - 6x = 6x(y-1), \quad f_y = 3x^2 + 3y^2 - 6y.$
  - **Step 1 – factor and split:** $6x(y-1) = 0$ holds only for $x = 0$ or $y = 1$.
  - **Step 2 – substitute each case into** $f_y = 0$, i.e. $x^2 + y^2 - 2y = 0$:

$$\begin{aligned} &x = 0: \ y^2 - 2y = y(y-2) = 0 \ \Rightarrow\ y = 0 \text{ or } y = 2 \\ &y = 1: \ x^2 - 1 = 0 \ \Rightarrow\ x = \pm 1 \end{aligned}$$

- **Critical points** (union of both cases): $(0,0),\ (0,2),\ (1,1),\ (-1,1)$.

Example 2 continued:

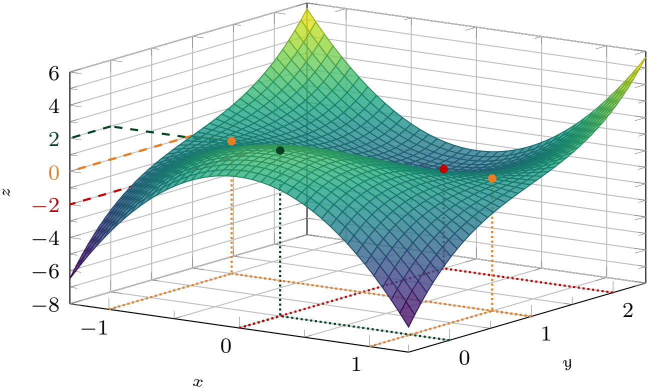
- SOCs at each of the critical points $(0,0)$, $(0,2)$, $(1,1)$, $(-1,1)$:

$$f_{xx} = 6y - 6, \qquad f_{yy} = 6y - 6, \qquad f_{xy} = f_{yx} = 6x$$

- at $(0,0)$: $f_{xx} = f_{yy} = -6$, $f_{xy} = 0$
  $\rightarrow$ $f_{xx}f_{yy} = 36 > 0$: **maximum**
- at $(0,2)$: $f_{xx} = f_{yy} = 6$, $f_{xy} = 0$
  $\rightarrow$ $f_{xx}f_{yy} = 36 > 0$: **minimum**
- at $(1,1)$: $f_{xx} = f_{yy} = 0$, $f_{xy} = 6$
  $\rightarrow$ $f_{xx}f_{yy} = 0 < 36$: **saddle point**
- at $(-1,1)$: $f_{xx} = f_{yy} = 0$, $f_{xy} = -6$
  $\rightarrow$ $f_{xx}f_{yy} = 0 < 36$: **saddle point**

Multivariable Functions (continued)

Example 2: $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$



Constrained Optimization

Simple problem: utility maximization

$$\text{Max/Min} \quad \underbrace{U = xy + 2x}_{\substack{\text{objective function} \\ \text{e.g., utility function}}} \qquad \text{subject to (s.t.)} \qquad \underbrace{4x + 2y = 60}_{\text{budget constraint}}$$

Note:

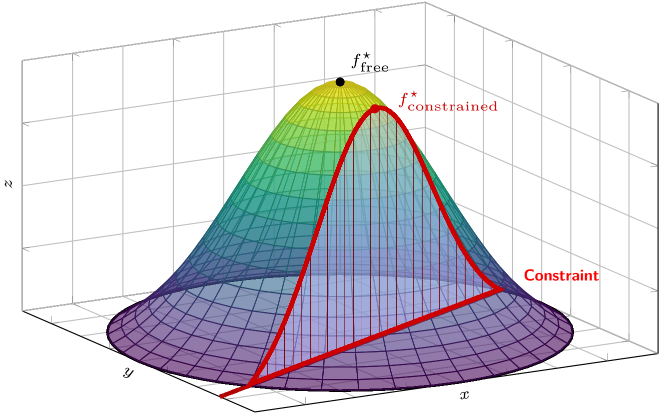
- $\frac{\partial U}{\partial x} > 0$ and $\frac{\partial U}{\partial y} > 0$ (for $x, y \in \mathbb{R}^+$) $\rightarrow$ to maximize U without constraints, the consumer needs to buy an *infinite* amount of both goods
- the budget constraint restricts the problem, taking into account the wealth of the consumer.

Simple solution:

- use the restriction to find $y = \dfrac{60 - 4x}{2} = 30 - 2x$
- substitute in $U = x(30 - 2x) + 2x = -2x^2 + 32x$
- solve the single-variable problem:
  - FOC: $\frac{\partial U}{\partial x} = -4x + 32 = 0 \rightarrow x = 8$ and $y = 14$
  - SOC: $\frac{\partial^2 U}{\partial x^2} = -4 < 0$: objective **maximized** at $(x^*, y^*) = (8, 14)$

Constrained Optimization

Illustration of a function with two variables $z = f(x, y)$



Schematic after Chiang (1984: 371)

The Lagrange multiplier method

- A solution to a problem

$$\text{Max/Min } f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) = c$$

- can be obtained by:
  1. Write down the Lagrangian function:

  $$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda\big(c - g(x_1, x_2)\big)$$

  where λ is the **Lagrange multiplier**.

  2. Differentiate $\mathcal{L}$ wrt x_1, x_2, and λ, and set all to 0 (FOCs):

  $$\frac{\partial \mathcal{L}}{\partial x_1} = 0, \qquad \frac{\partial \mathcal{L}}{\partial x_2} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = c - g(x_1, x_2) = 0$$

  3. Solve for the three unknowns x_1, x_2, λ to find the critical values.

Lagrange multiplier method — Example 1:

$\max U = xy + 2x$ s.t. $4x + 2y = 60$

1. Write the Lagrangian: $\mathcal{L}(x, y, \lambda) = xy + 2x + \lambda(60 - 4x - 2y)$
2. Differentiate wrt x, y, λ and set to 0:

$$\frac{\partial \mathcal{L}}{\partial x} = y + 2 - 4\lambda = 0 \quad (1), \quad \frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda = 0 \quad (2), \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 60 - 4x - 2y = 0 \quad (3)$$

3. Solve for x, y, λ:
   - from 1st eq.: $\lambda = \frac{y}{4} + \frac{1}{2}$; from 2nd eq.: $\lambda = \frac{x}{2}$

   $\rightarrow$ thus $\frac{y}{4} + \frac{1}{2} = \frac{x}{2} \Leftrightarrow x = \frac{y}{2} + 1$.
   - in 3rd eq.: $4(\frac{y}{2} + 1) + 2y = 60 \Rightarrow y = 14$
   - then $x = 8$, and from 1st/2nd eq.: $\lambda = 4$.

$\Rightarrow$ **Critical values**: $(x^*, y^*, \lambda^*) = (8, 14, 4)$

Significance of the Lagrange multiplier

- The Lagrange multiplier λ approximates the **marginal impact on the objective function** caused by a small change in the constraint:

$\rightarrow$ a 1-unit increase (decrease) in c would cause the optimal value to increase (decrease) by approximately λ units.

  - e.g., $U_{\max}$ should increase by 4 units if the constraint changes from $4x + 2y = 60$ to $4x + 2y = 61$

  $\rightarrow$ Economic interpretation: when the budget constraint changes by 1 CHF, utility changes by 4 units $=$ the marginal utility of 1 CHF

- λ is referred to as the **shadow price** of the resource

- NB: $\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda(c - g(x_1, x_2))$ is equivalent to

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(g(x_1, x_2) - c)$$

  - critical values are identical
  - only the sign of λ differs

Lagrange multiplier method — Example 2:

Optimize $f(x,y) = 4x^2 + 3xy + 6y^2$ s.t. $g(x,y) = x + y = 56$

Set up the Lagrangian:

$$\mathcal{L}(x,y,\lambda) = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$

Differentiate wrt x, y, λ and set to zero:

$$\frac{\partial \mathcal{L}}{\partial x} = 8x+3y-\lambda = 0 \ \ (1), \qquad \frac{\partial \mathcal{L}}{\partial y} = 3x+12y-\lambda = 0 \ \ (2), \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = 56-x-y = 0 \ \ (3)$$

Solve for x, y, λ:

- subtract (2) from (1): $5x - 9y = 0 \Rightarrow x = \frac{9}{5}y$
- substitute into (3): $56 - \frac{9}{5}y - y = 0 \Rightarrow y = 20 \Rightarrow x = 36 \Rightarrow \lambda = 348$

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Critical values $(x^*, y^*, \lambda^*) = (36, 20, 348)$ $\Longrightarrow$ **Min or Max?**

Constrained Optimization — Key building blocks

1. The Hessian of a function is the (symmetric) matrix of its second-order partial derivatives. For $f(x, y)$:

$$\mathbf{H}(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad (f_{xy} = f_{yx} \text{ when the partials are continuous, by Young})$$

2. The bordered Hessian is simply the *full* Hessian of the Lagrangian $\mathcal{L}(x, y, \lambda)$ including second derivatives w.r.t. *all* variables, with λ listed first:

$$\overline{\mathbf{H}} = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{\lambda y} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$

since $\mathcal{L}_{\lambda\lambda} = 0$ and $\mathcal{L}_{\lambda x} = -g_x,\ \mathcal{L}_{\lambda y} = -g_y$. The "border" is just the constraint g's gradient.

Constrained Optimization — Computing the determinant

3. The determinant of a 2×2 matrix is the product of the main diagonal minus the product of the anti-diagonal:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The 3×3 determinant is built from 2×2 ones by *cofactor expansion* along a row (or column): each row-1 entry is multiplied by the 2×2 *minor* left after deleting its row and column (with alternating signs $+ - +$).

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{aligned} \det = {} & a\,(ei - fh) \\ & - b\,(di - fg) \\ & + c\,(dh - eg) \end{aligned}$$

Step 1: entry a, minor $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$

Constrained Optimization — Computing the determinant

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$$\begin{vmatrix} a & \boxed{b} & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{aligned} \det = {} & a\,(ei - fh) \\ & - b\,(di - fg) \\ & + c\,(dh - eg) \end{aligned}$$

Step 2: entry b, minor $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$

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$$\begin{vmatrix} a & b & \boxed{c} \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\begin{aligned} \det &= a\,(ei - fh) \\ &\quad - b\,(di - fg) \\ &\quad + c\,(dh - eg) \end{aligned}$$

Step 3: entry c, minor $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Constrained Optimization (continued)

Lagrange multiplier method

Is $(36, 20, 348)$ a minimum or a maximum?

Bordered Hessian test

$|\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| > 0 \;\Rightarrow\;$ local maximum $\qquad |\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| < 0 \;\Rightarrow\;$ local minimum

$$|\overline{\mathbf{H}}| = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix} = -(-g_x)\left[-g_x \mathcal{L}_{yy} - \mathcal{L}_{xy}(-g_y)\right] + (-g_y)\left[-g_x \mathcal{L}_{yx} - \mathcal{L}_{xx}(-g_y)\right]$$

NB: $\overline{\mathbf{H}}$ is the bordered Hessian *matrix*; $|\overline{\mathbf{H}}|$ is its determinant.

Lagrange multiplier method

Second-order partials: direct $\mathcal{L}_{xx} = 8$, $\mathcal{L}_{yy} = 12$; cross $\mathcal{L}_{xy} = 3 = \mathcal{L}_{yx}$

$$g(x, y) = x + y = 56 \;\Rightarrow\; g_x = 1,\; g_y = 1$$

Calculate the determinant of the bordered Hessian:

$$|\overline{\mathbf{H}}(x^*, y^*, \lambda^*)| = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 8 & 3 \\ -1 & 3 & 12 \end{vmatrix}$$

$$= -(-1)\Big((-1) \times 12 - (-1) \times 3\Big) + (-1)\Big((-1) \times 3 - (-1) \times 8\Big)$$

$$= -9 + (-5) = -14 < 0.$$

Result

$(x^*, y^*, \lambda^*) = (36, 20, 348)$ is a **local minimum**.

Constrained Optimization (continued)

Lagrange multiplier method — back to the first example

- Compute the local max/min of $f(x,y) = xy + 2x$ s.t. $g(x,y) = 4x + 2y = 60$

$$\mathcal{L}(x,y,\lambda) = xy + 2x + \lambda(60 - 4x - 2y)$$

$\rightarrow$ Critical value: $(x^*, y^*, \lambda^*) = (8, 14, 4)$

- First-order partials:

$$\mathcal{L}_x = y + 2 - 4\lambda \qquad \mathcal{L}_y = x - 2\lambda$$

- Second-order partials: direct $\mathcal{L}_{xx} = 0$, $\mathcal{L}_{yy} = 0$; cross $\mathcal{L}_{xy} = 1 = \mathcal{L}_{yx}$

$$g(x,y) = 4x + 2y = 60 \;\Rightarrow\; g_x = 4,\; g_y = 2$$

Lagrange multiplier method

Calculate the determinant of the bordered Hessian:

$$\overline{\mathbf{H}}(x^*, y^*, \lambda^*) = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ -g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -4 & -2 \\ -4 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} = 16 > 0$$

Result

$(x^*, y^*, \lambda^*) = (8, 14, 4)$ is a **local maximum**.

Multi-constraint case

- Consider a problem with n variables and m constraints:

$$\underset{x_1,\ldots,x_n}{\mathrm{Max/Min}}\, f(x_1,\ldots,x_n) \quad \text{s.t.} \quad \begin{cases} g_1(x_1,\ldots,x_n) = c_1 \\ \quad\vdots \\ g_m(x_1,\ldots,x_n) = c_m \end{cases}$$

- The Lagrangian is:

$$\mathcal{L}(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m) = f(x_1,\ldots,x_n) + \sum_{j=1}^{m} \lambda_j (c_j - g_j(x_1,\ldots,x_n))$$

- The FOCs write:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \lambda_j} = c_j - g_j(x_1,\ldots,x_n) = 0$$

Constrained Optimization (continued)

Example: multi-constraint case

Optimize $f(x,y,z) = x^2 + y^2 + z^2$ s.t.

$$\begin{cases} g_1(x,y,z) = x + y + z = 1 \\ g_2(x,y,z) = x + 2y + 3z = 6 \end{cases}$$

- **Step 1**: set up the Lagrangian

$$\mathcal{L}(x,y,z,\lambda_1,\lambda_2) = x^2 + y^2 + z^2 + \lambda_1(1 - x - y - z) + \lambda_2(6 - x - 2y - 3z)$$

- **Step 2**: take the first partial derivatives wrt $x, y, z, \lambda_1, \lambda_2$:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda_1 - \lambda_2 = 0 \qquad \frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda_1 - 2\lambda_2 = 0 \qquad \frac{\partial \mathcal{L}}{\partial z} = 2z - \lambda_1 - 3\lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = 1 - x - y - z = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda_2} = 6 - x - 2y - 3z = 0$$

Solving the 5×5 Lagrange System

Idea: the stationarity equations are linear, so solve for x, y, z in terms of λ_1, λ_2, then feed them into the two constraints.

1. **Solve the first three equations for x, y, z:**

$$x = \frac{\lambda_1 + \lambda_2}{2}, \qquad y = \frac{\lambda_1 + 2\lambda_2}{2}, \qquad z = \frac{\lambda_1 + 3\lambda_2}{2}.$$

2. **Substitute into the constraints** $x + y + z = 1$ and $x + 2y + 3z = 6$:

$$3\lambda_1 + 6\lambda_2 = 2, \qquad 3\lambda_1 + 7\lambda_2 = 6.$$

3. **Solve the 2×2 system** (subtract the first from the second):

$$\lambda_2 = 4, \qquad \lambda_1 = -\frac{22}{3}.$$

4. **Back-substitute** to recover the variables:

$$x = -\frac{5}{3}, \qquad y = \frac{1}{3}, \qquad z = \frac{7}{3}.$$

Check: $x + y + z = 1$ ✓ $\qquad x + 2y + 3z = 6$ ✓

Example: multi-constraint case (continued)

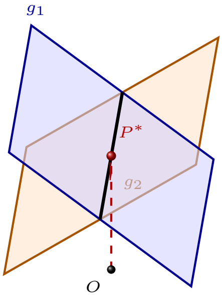
Solving the five-equation system yields the critical values:

Critical point

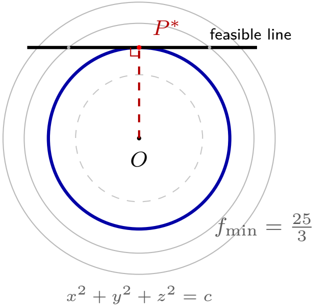
$$(x^*, y^*, z^*) = \left(-\tfrac{5}{3}, \tfrac{1}{3}, \tfrac{7}{3}\right), \qquad (\lambda_1^*, \lambda_2^*) = \left(-\tfrac{22}{3}, 4\right)$$

- The objective $f(x, y, z) = x^2 + y^2 + z^2$ is the squared distance from the origin, so this point is the **minimum**-distance point on the intersection of the two constraint planes.

Isosurfaces of $f(x, y, z) = x^2 + y^2 + z^2 = c$ are *spheres* in the domain $\mathbb{R}^3$. The minimum is the smallest sphere that still touches the feasible line.



Feasible set $= g_1 \cap g_2$ (a line in $\mathbb{R}^3$)



Isosurfaces of f (spheres), sliced through O; smallest is tangent at P^*