Math Camp — Day 1 Exercises

Worked Solutions

Exercise 1 — Differentiation

a) $f(x) = 18x^{1/2}$**.** Power rule:

$$f'(x) = 18 \cdot \tfrac{1}{2}x^{-1/2} = 9x^{-1/2} = \frac{9}{\sqrt{x}}.$$

b) $f(t) = (2t^4+5)(3t^5-8)$**.** Product rule with $u = 2t^4+5,\ v = 3t^5-8\ (u' = 8t^3,\ v' = 15t^4)$:

$$f'(t) = 8t^3(3t^5-8) + (2t^4+5)15t^4 = 24t^8 - 64t^3 + 30t^8 + 75t^4 = 54t^8 + 75t^4 - 64t^3.$$

c) $f(x) = \dfrac{10x^8 - 6x^7}{2x}$**.** Simplify first:

$$f(x) = \frac{10x^8}{2x} - \frac{6x^7}{2x} = 5x^7 - 3x^6 \quad \Longrightarrow \quad f'(x) = 35x^6 - 18x^5.$$

d) $f(y) = (6y^3+9)^4$**.** Chain rule (inner derivative $18y^2$):

$$f'(y) = 4(6y^3+9)^3 \cdot 18y^2 = 72y^2(6y^3+9)^3.$$

e) $f(x) = \dfrac{1}{7x^3+13x+3} = (7x^3+13x+3)^{-1}$**.** Chain rule (inner derivative $21x^2+13$):

$$f'(x) = -(7x^3+13x+3)^{-2}(21x^2+13) = -\frac{21x^2+13}{(7x^3+13x+3)^2}.$$

f) $f(u) = \left(\dfrac{3u+4}{2u+5}\right)^2$**.** Let $g = \dfrac{3u+4}{2u+5}$. Quotient rule:

$$g' = \frac{3(2u+5) - (3u+4)2}{(2u+5)^2} = \frac{6u+15-6u-8}{(2u+5)^2} = \frac{7}{(2u+5)^2}.$$

Then $f' = 2g\,g'$:

$$f'(u) = 2 \cdot \frac{3u+4}{2u+5} \cdot \frac{7}{(2u+5)^2} = \frac{14(3u+4)}{(2u+5)^3}.$$

Exercise 2 — First/second derivatives; behaviour at $x = 2$

a) $f(x) = 7x^3 + 5x^2 + 12$**.**

$$f'(x) = 21x^2 + 10x, \qquad f''(x) = 42x + 10.$$

At $x = 2$: $f'(2) = 84 + 20 = 104 > 0$ (**increasing**); $f''(2) = 84 + 10 = 94 > 0$ (**convex**).

b) $f(x) = (x^4 - 3)(x^3 - 2) = x^7 - 2x^4 - 3x^3 + 6$**.**

$$f'(x) = 7x^6 - 8x^3 - 9x^2, \qquad f''(x) = 42x^5 - 24x^2 - 18x.$$

At $x = 2$: $f'(2) = 448 - 64 - 36 = 348 > 0$ (**increasing**); $f''(2) = 1344 - 96 - 36 = 1212 > 0$ (**convex**).

c) $f(x) = \dfrac{7x^2}{x-1}$**.** Quotient rule:

$$f'(x) = \frac{14x(x-1) - 7x^2}{(x-1)^2} = \frac{7x^2 - 14x}{(x-1)^2} = \frac{7x(x-2)}{(x-1)^2}.$$

Differentiating again and cancelling one factor $(x-1)$ gives

$$f''(x) = \frac{14}{(x-1)^3}.$$

At $x = 2$: $f'(2) = 0$ (**stationary**); $f''(2) = 14 > 0$ (**convex**) — so a local minimum.

Exercise 3 — Second-order and cross partials

a) $f(x_1, x_2) = x_1^3 - 9x_1x_2 - 3x_2^2$**.**

$$f_1 = 3x_1^2 - 9x_2, \qquad f_2 = -9x_1 - 6x_2.$$

$$f_{11} = 6x_1, \qquad f_{22} = -6, \qquad f_{12} = f_{21} = -9.$$

b) $f(x_1, x_2) = (7x_1 + 3x_2)^3$**.**

$$f_1 = 21(7x_1 + 3x_2)^2, \qquad f_2 = 9(7x_1 + 3x_2)^2.$$

$$f_{11} = 294(7x_1 + 3x_2), \quad f_{22} = 54(7x_1 + 3x_2), \quad f_{12} = f_{21} = 126(7x_1 + 3x_2).$$

(Cross partials agree, confirming Young's theorem.)

Exercise 4 — Logarithmic functions

a) $f(x) = \ln(2x^3) = \ln 2 + 3\ln x$**.**

$$f'(x) = \frac{3}{x}.$$

b) $f(x) = (\ln 8x)^2$**.** Chain rule ($\frac{d}{dx}\ln 8x = \frac{1}{8x} \times 8 = \frac{1}{x}$):

$$f'(x) = 2(\ln 8x) \cdot \frac{1}{x} = \frac{2\ln 8x}{x}.$$

c) $f(x) = \ln \dfrac{x^3}{(2x+5)^2} = 3\ln x - 2\ln(2x+5)$**.**

$$f'(x) = \frac{3}{x} - 2 \cdot \frac{2}{2x+5} = \frac{3}{x} - \frac{4}{2x+5} = \frac{2x+15}{x(2x+5)}.$$

Exercise 5 — Exponential functions

a) $f(x) = e^{2x}$**.** $\quad f'(x) = 2e^{2x}$.

b) $f(x) = x^2 e^{5x}$**.** Product rule:

$$f'(x) = 2xe^{5x} + x^2 \cdot 5e^{5x} = xe^{5x}(2 + 5x).$$

c) $f(x) = \dfrac{e^{5x} - 1}{e^{5x} + 1}$**.** Quotient rule ($u' = v' = 5e^{5x}$):

$$f'(x) = \frac{5e^{5x}(e^{5x} + 1) - (e^{5x} - 1)5e^{5x}}{(e^{5x} + 1)^2} = \frac{5e^{5x}\left[\,(e^{5x} + 1) - (e^{5x} - 1)\,\right]}{(e^{5x} + 1)^2} = \frac{10e^{5x}}{(e^{5x} + 1)^2}.$$

Exercise 6 — Definite integrals

a) $\displaystyle\int_0^6 5x\,\mathrm{d}x$**.**

$$= \left[\frac{5x^2}{2}\right]_0^6 = \frac{5 \cdot 36}{2} = 90.$$

b) $\displaystyle\int_0^{10} 2e^{-2x}\,\mathrm{d}x$**.**

$$= \left[-e^{-2x}\right]_0^{10} = -e^{-20} + e^0 = 1 - e^{-20} \approx 1.$$

c) $\displaystyle\int_0^3 8x(2x^2 + 3)\,\mathrm{d}x$**.**

$$= \int_0^3 (16x^3 + 24x)\,\mathrm{d}x = [4x^4 + 12x^2]_0^3 = 324 + 108 = 432.$$

Alternatively (substitution method):

Substitute $u = 2x^2 + 3$, $\mathrm{d}u/\mathrm{d}x = 4x \Rightarrow \mathrm{d}u = 4x\,\mathrm{d}x$, so $8x\,\mathrm{d}x = 2\,\mathrm{d}u$;
limits $x{=}0 \to u{=}3$, $x{=}3 \to u{=}21$:

$$= \int_3^{21} 2u\,\mathrm{d}u = \left[u^2\right]_3^{21} = 441 - 9 = 432.$$