Refresher Course in Calculus, Probability, and Statistics

Day 1: Calculus

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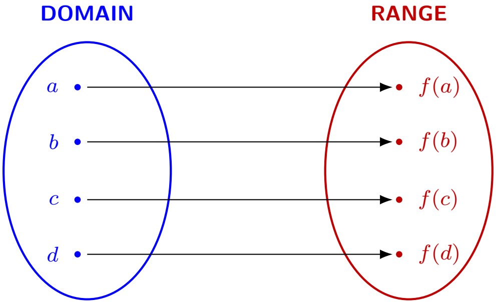
Function

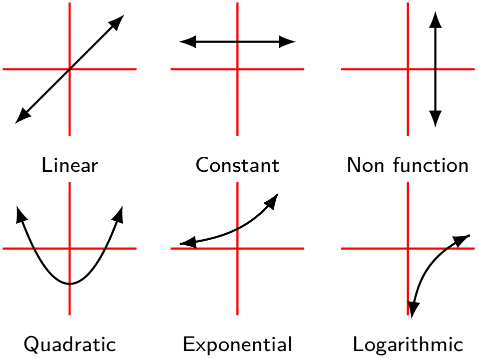
- Definition:
  - A function of a real variable x with domain D is a rule that assigns a unique real number to each number x in D.
  - Given a function f, the set of numbers x at which $f(x)$ is defined is called the **domain** of f. The set of values $f(x)$ obtained as x varies in the domain is called the **range** of f.



  - Example: a supply function $y = a + bp$, where y is output, p is price.

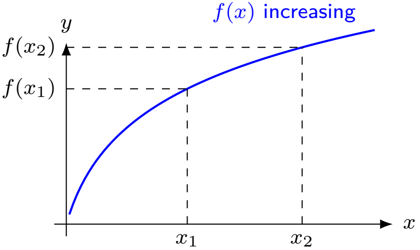
Function Forms

- Examples:
  - Linear function: $y = ax + b$
  - Quadratic function: $y = ax^2 + bx + c$
  - Natural exponential function: $y = e^x$
  - Natural logarithmic function: $y = \ln x$



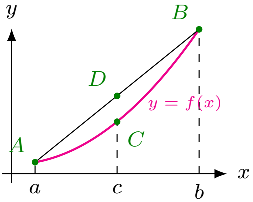
Geometric Properties of a Function

- Increasing or decreasing

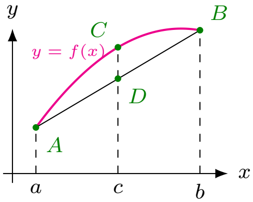


- Concave or convex

Convex



Concave



Introduction

- **Differential calculus** is the study of the rate at which quantities change.
  - The primary objects of study are the *derivative* and the *primitive function*.
  - The process of finding a derivative is called *differentiation*.
- **Integral calculus** can be regarded as the "inverse" of differentiation.
  - The process of reconstructing a function from its derivative is called *integration*.
  - Two kinds of integrals:
    - indefinite integral $\rightarrow$ a function
    - definite integral $\rightarrow$ a number

$\Rightarrow$ Focus on differential calculus and know how to get a derivative.

- **References:**
  - Paul's Online Math Notes: `http://tutorial.math.lamar.edu/`

Difference Quotient

- A change in a variable x is denoted: $\Delta x = x_1 - x_0$
- Corresponding change of the function $y = f(x)$:

$$\Delta y = f(x_1) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

- **Difference quotient**: change in y per unit of change in x:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$\rightarrow$ Expresses the *average* change in y for $\Delta x = 1$ on $[x_0, x_1]$

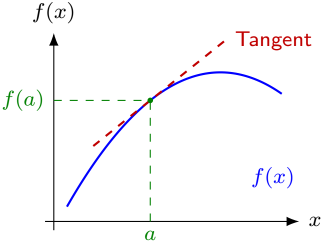
Derivative of a function: Formal Definitions

For a function $y = f(x)$

- At point x_0:

$$f'(x_0) \;=\; \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

  - $f'(x_0)$ is the *instantaneous* / *marginal* rate of change
  - $f' \geq 0 \; (> 0) \iff f$ increasing (strictly increasing)
  - $f' \leq 0 \; (< 0) \iff f$ decreasing (strictly decreasing)



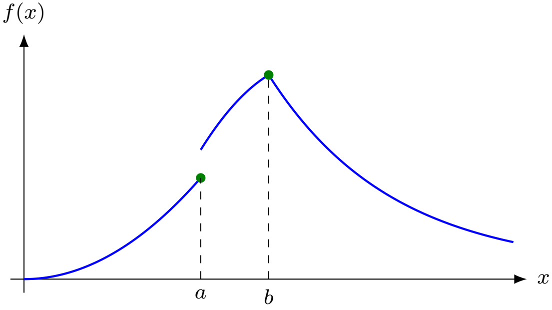
- At any point x:

$$f'(x) \;=\; \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$f'(x)$ is a function describing the *instantaneous* rate of change of $y = f(x) \; \forall x$.

Differentiability and Continuity

- A function is said to be **differentiable** at a point if its derivative exists at that point.
- To be differentiable at a point, a function must:
  - be continuous **AND**
  - have a unique tangent



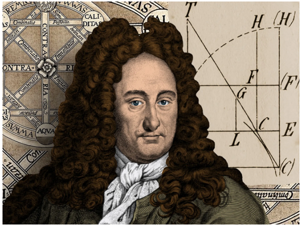
- Not a differentiable function: discontinuity at a and non-differentiable corner at b.

Notation of the Derivative

- The derivative of $y = f(x)$ may be written in several *equivalent* ways:

$$f'(x) \qquad y' \qquad f_x(x) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} \qquad \frac{\mathrm{d}f(x)}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

- Leibniz's notation (with d's)
  - preferred in economics
  - emphasises that $f(x)$ is differentiated with respect to (wrt) x
  - can prevent confusion for functions of several variables $f(x_1, \ldots, x_n)$ and for higher-order derivatives



Gottfried Wilhelm Leibniz (1646–1716)

Basic Rules for Differentiation

Function $f(x)$	**Derivative** $\dfrac{\mathrm{d}f(x)}{\mathrm{d}x}$
α	0
αx	α
x^α	$\alpha x^{\alpha-1}$
α^x	$\alpha^x \ln \alpha$
e^x	e^x
$\ln x$	$\dfrac{1}{x}$

Simple Examples

- Compute $f'(x)$ when $f(x) = x^3$:
  - Using the definition (let $\Delta x = h$):

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2 \end{aligned}$$

  - Using the power rule $f(x) = x^\alpha \Rightarrow \dfrac{\mathrm{d}f(x)}{\mathrm{d}x} = \alpha x^{\alpha-1}$: $\quad \dfrac{\mathrm{d}x^3}{\mathrm{d}x} = 3x^2$

- Compute $f'(x)$ when $f(x) = \dfrac{1}{x}$ (note: $\frac{1}{x} = x^{-1}$)

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -1\,x^{-2} = -\frac{1}{x^2}$$

Differentiation of Sums, Products, and Quotients

Function	**Derivative**
$h(x)$	$\dfrac{\mathrm{d}h(x)}{\mathrm{d}x}$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

- **Composite function** (or function of a function): a function in which y is a function of u and u in turn is a function of x.
- Derivative of y wrt x:

$$y = f(u) \text{ and } u = g(x) \quad \Longrightarrow \quad y' = f'(u) \cdot g'(x) = f'[g(x)] \cdot g'(x)$$

$$\text{or:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

- Example: $f(x) = \left(1 - x^3\right)^5$, with $g(x) = 1 - x^3$ and $f(u) = u^5$:

$$f'(x) = 5\left(1 - x^3\right)^4 \cdot \left(-3x^2\right) = -15x^2\left(1 - x^3\right)^4$$

Higher-Order Derivatives

- $f'(x)$: **first derivative** of f
- $f''(x)$: **second-order derivative** of f

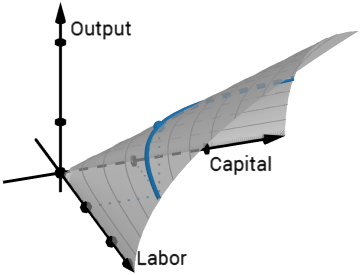
$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$$

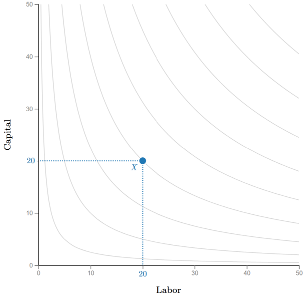
  - $f''(x) > 0 \iff f$ convex (at x)
  - $f''(x) < 0 \iff f$ concave (at x)

- $f'''(x) = \dfrac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3}$: **third-order derivative** of f
- $f''''(x) = \dfrac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4}$: **fourth-order derivative** of f
- $\ldots$

Multi-variable Functions

- With functions of several variables, one can measure the rate of change of y wrt x_1 while x_2 is held constant.
- NB: In economics, "*held constant*" is used equivalently to *"all other things being equal"* or *"ceteris paribus"*.





Cobb-Douglas production function: $Y = f(K, L) = AL^aK^b$ with $A = 1,\ a = 0.5,\ b = 0.5.$

Set your own parameters at:
`https://www.econgraphs.org/graphs/firm/technology/cobb_douglas_production_3d`

Multivariable Functions and Partial Derivatives

- For functions of several variables $y = f(x_1, x_2, \dots)$, the (first-order) **partial derivative**:
  - is defined as $\frac{\partial y}{\partial x_1}$: (first-order) partial derivative of $f(x_1, x_2, \dots)$ with respect to x_1
  - ∂ (read: "(partial) dee" or "partial") has the same function as d in *Leibniz notation*
  - measures the rate of change of y wrt x_1 while x_2 is held constant ("*ceteris paribus*")
- Notations: $\frac{\partial y}{\partial x_1}, \quad \frac{\partial f}{\partial x_1}, \quad f_1(x_1, x_2, \dots), \quad f_{x_1}(x_1, x_2, \dots), \quad y_1$
- Partial differentiation wrt one of the independent variables follows the **same rules** as ordinary differentiation, with the other variables **treated as constants**.

Second-Order Partial Derivatives

- Second-order **direct** partial derivatives of $y = f(x_1, x_2)$:

$$\text{wrt } x_1 : \; f_{11} = f_{x_1 x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_1}\right) = \frac{\partial^2 y}{\partial x_1^2}$$

$$\text{wrt } x_2 : \; f_{22} = f_{x_2 x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_2^2}$$

- **Cross (or mixed)** partial derivatives:

$$f_{12} = f_{x_1 x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_1 \partial x_2}$$

$$f_{21} = f_{x_2 x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\right) = \frac{\partial^2 y}{\partial x_2 \partial x_1}$$

  - If both cross partial derivatives are continuous, they are identical: $f_{12} = f_{21}$

$\rightarrow$ *Young's theorem*.

Exponential Functions

- Exponential function:

$$f(x) = a^x, \qquad a > 0 \qquad (a : \text{base},\ x : \text{exponent})$$

- **Properties**:
  - Same base:
    $$a^x \cdot a^y = a^{x+y}$$
    $$\frac{a^x}{a^y} = a^{x-y}$$
    $$(a^x)^y = a^{x \cdot y}$$
  - Same exponent:
    $$a^x \cdot b^x = (ab)^x$$
    $$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

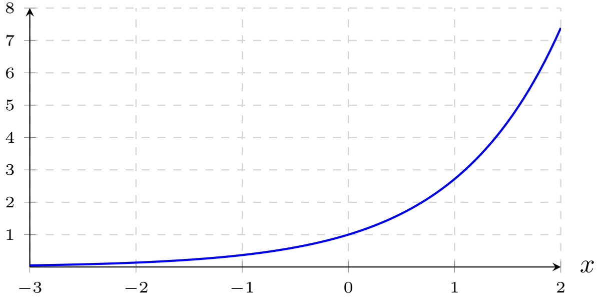
- **Derivative**: $f'(x) = a^x \ln a$

Natural Exponential Function

- **Natural exponential** function:

$$f(x) = e^x = \exp(x), \quad \text{with } e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828\ldots, \text{ known as } \textit{“Euler's number”}$$

- **Derivatives**: $\quad f'(x) = e^x, \quad f''(x) = e^x, \quad \ldots$



Logarithmic Functions

- Logarithmic function with base a:

$$f(x) = \log_a x, \qquad a > 0,\ a \neq 1 \qquad (a : \text{base},\ \log_a x : \text{exponent})$$

Definition: $\log_a x$ is the exponent to which a must be raised to obtain x

$$x = a^y \iff y = \log_a x$$

- Properties:
  - $a^{\log_a x} = \log_a a^x = x$
  - $\log_a(x \cdot y) = \log_a x + \log_a y$
  - $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
  - $\log_a x^n = n \cdot \log_a x$

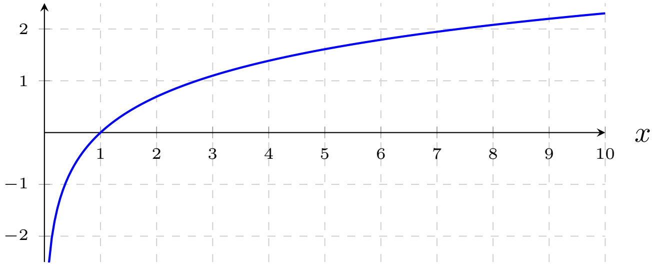
    $\iff \log_a \sqrt[n]{x} = \log_a x^{\frac{1}{n}} = \frac{1}{n} \cdot \log_a x$
  - $\log_a x = (\log_a c)(\log_c x)$

    $\iff \log_c x = \frac{\log_a x}{\log_a c}$

- Derivative: $\quad f'(x) = \dfrac{1}{x \ln a}$

Natural Logarithmic Function

- Natural logarithmic function: $f(x) = \log_e x = \ln x$
- Inverse function of e^x: $e^{\ln x} = \ln(e^x) = x$
- Derivatives: $f'(x) = \dfrac{1}{x}, \quad f''(x) = -\dfrac{1}{x^2}, \quad f'''(x) = \dfrac{2}{x^3}, \quad \ldots$



Logarithmic Differentiation

- Logarithmic differentiation: $\dfrac{\mathrm{d} \ln f(x)}{\mathrm{d}x} = \dfrac{f'(x)}{f(x)}$ iff $f(x)$ is differentiable

  Demonstration

  - Suppose that $y = \ln f(x)$ and $f(x)$ is differentiable
  - This is a composite function $y = g[f(x)]$, such that $g(u) = \ln u$ and $u = f(x)$
  - Differentiate both functions wrt u and x respectively:
    $g' = \frac{1}{u}$ and $u' = f'(x)$
  - Apply the chain rule $y' = g'(u) \cdot f'(x) \quad \Rightarrow \quad y' = g' \cdot u' = \frac{1}{u} f'(x) = \frac{f'(x)}{u}$.
  - Recall that $u = f(x)$ and substitute:

$$y' = \frac{f'(x)}{u} = \frac{f'(x)}{f(x)}$$

- Use this property to differentiate awkward functions:

$$\boxed{f'(x) = \frac{\mathrm{d} f(x)}{\mathrm{d}x} = \frac{\mathrm{d} \ln f(x)}{\mathrm{d}x} f(x)}$$

Logarithmic Differentiation (continued)

Example 1: find $f'(x)$

- $f(x) = x^x$
  - Taking the logarithm:

$$\ln f(x) = \ln x^x = x \ln x$$

  - Differentiate both sides wrt x (using logarithmic differentiation on the lhs):

$$\frac{f'(x)}{f(x)} = 1 \cdot \ln x + x \left(\frac{1}{x} \right) = \ln x + 1$$

  - Multiply both sides by $f(x)$:

$$f'(x) = (\ln x + 1)\, x^x$$

Logarithmic Differentiation (continued)

Example 2: find $f'(x)$

- $f(x) = A(x)^\alpha B(x)^\beta$
  - Taking the logarithm:

$$\ln f(x) = \ln\big(A(x)^\alpha B(x)^\beta\big) = \alpha \ln A(x) + \beta \ln B(x)$$

  - Differentiate both sides wrt x (using logarithmic differentiation):

$$\frac{f'(x)}{f(x)} = \alpha \frac{1}{A(x)} A'(x) + \beta \frac{1}{B(x)} B'(x) = \alpha \frac{A'(x)}{A(x)} + \beta \frac{B'(x)}{B(x)}$$

  - Multiply both sides by $f(x)$:

$$f'(x) = \left(\alpha \frac{A'(x)}{A(x)} + \beta \frac{B'(x)}{B(x)}\right) A(x)^\alpha B(x)^\beta$$

Elasticities

- Elasticity of $y = f(x)$ with respect to x
  - ratio between the relative change in y and the relative change in x
  - dimensionless measure of association

$$\varepsilon_{yx} = \frac{\frac{\mathrm{d}y}{y}}{\frac{\mathrm{d}x}{x}} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{x}{y}$$

  - $f(x)$ is *elastic* at a point when $\varepsilon_{yx} > 1$
    of unit elasticity at a point when $\varepsilon_{yx} = 1$
    inelastic at a point when $\varepsilon_{yx} < 1$

- Generally: $\varepsilon_{yx} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{x}{y} = \frac{\mathrm{d}\ln y}{\mathrm{d}\ln x}$ because $\mathrm{d}\ln y = \frac{\mathrm{d}y}{y}$ and $\mathrm{d}\ln x = \frac{\mathrm{d}x}{x}$
- Partial elasticities for functions of several variables $y = f(x_1, \ldots, x_n)$ are defined similarly:

$$\varepsilon_{yx_1} = \frac{\partial y}{\partial x_1} \cdot \frac{x_1}{y}, \quad \ldots, \quad \varepsilon_{yx_n} = \frac{\partial y}{\partial x_n} \cdot \frac{x_n}{y}$$

Elasticities (continued)

Example: find the elasticity of Y with respect to K for $Y = AK^\alpha L^\beta$

- Definition of elasticity: $\varepsilon_{YK} = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y}$
  - $\frac{\partial Y}{\partial K}$ is the partial derivative of Y wrt K: $\frac{\partial Y}{\partial K} = \alpha A K^{\alpha-1} L^\beta$
  - $\frac{K}{Y}$ is simply the ratio of K and Y: $\frac{K}{Y} = \frac{K}{AK^\alpha L^\beta} = \frac{1}{AK^{\alpha-1}L^\beta}$
  - Combining the two terms:
  $$\varepsilon_{YK} = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = \alpha A K^{\alpha-1} L^\beta \cdot \frac{1}{AK^{\alpha-1}L^\beta} = \alpha.$$
- Alternative: log transformation
  - $\ln Y = \ln\left(AK^\alpha L^\beta\right) = \ln A + \alpha \ln K + \beta \ln L$
  - Taking the partial derivative of $\ln Y$ wrt $\ln K$: $\frac{\partial \ln Y}{\partial \ln K} = \alpha.$

Derivatives (further information and exercises)

- More comprehensive discussion of differentiation (on Paul Dawkins' website)
  - One-variable derivatives:
    - Explanation:
      `https://tutorial.math.lamar.edu/Classes/CalcI/DerivativeIntro.aspx`
    - Exercises:
      `https://tutorial.math.lamar.edu/Problems/CalcI/DerivativeIntro.aspx`
  - Partial derivatives:
    - Explanation:
      `https://tutorial.math.lamar.edu/Classes/CalcIII/PartialDerivsIntro.aspx`
    - Exercises:
      `https://tutorial.math.lamar.edu/Problems/CalcIII/PartialDerivsIntro.aspx`
  - Applications:
    - Explanation:
      `https://tutorial.math.lamar.edu/Classes/CalcI/DerivAppsIntro.aspx` &
      `https://tutorial.math.lamar.edu/Classes/CalcIII/PartialDerivAppsIntro.aspx`
    - Exercises:
      `https://tutorial.math.lamar.edu/Problems/CalcI/DerivAppsIntro.aspx` &
      `https://tutorial.math.lamar.edu/Problems/CalcIII/PartialDerivAppsIntro.aspx`

Indefinite Integral

- Suppose we do not know the primitive function H.
  - All we know is its derivative: $H'(t) = \dfrac{\mathrm{d}H}{\mathrm{d}t} = t^{-\frac{1}{2}}$
  - We can use the differentiation rule: $f(x) = x^\alpha \Rightarrow f'(x) = \alpha x^{\alpha-1}$
  - And reverse it: $f'(x) = x^\alpha \Rightarrow f(x) = \dfrac{1}{\alpha+1}x^{\alpha+1}$
- Possible solution for H:
  - $H(t) = 2t^{\frac{1}{2}}$ ...but what about $H(t) = 2t^{\frac{1}{2}} + 99$ or $H(t) = 2t^{\frac{1}{2}} + 27000$?
- Because all constants disappear, the indefinite integral $H(t) = 2t^{\frac{1}{2}} + c$ is a *group* of functions, where c is an arbitrary constant
  - To solve for H explicitly we need additional information:
    an *initial* or *boundary condition* (the state of H at a given value of t), e.g:
    $H(0) = 33 \;\Rightarrow\; 33 = 2 \cdot 0^{\frac{1}{2}} + c \Longleftrightarrow c = 33.$
  - $\Rightarrow$ $H(t) = 2t^{\frac{1}{2}} + 33$

Indefinite Integral

- Definition of the **indefinite integral**:

$$\int f(x)\,\mathrm{d}x = F(x) + c \qquad \text{when } F'(x) = f(x)$$

- Basic vocabulary:
  - symbol $\int$: integral sign
  - function $f(x)$: integrand
  - x: variable of integration (indicated by $\mathrm{d}x$)
  - c: constant of integration
- We say *indefinite* because the technique of integration is not applied to any numbers, borders, or limits

→ the result is usually a *function*.

Common Integrals

Function	Anti-derivative
$\int x^\alpha \, \mathrm{d}x$	$\frac{1}{\alpha+1} x^{\alpha+1} + c \quad \text{if } \alpha \neq -1$
$\int \alpha \, \mathrm{d}x = \int \alpha x^0 \, \mathrm{d}x$	$\alpha x + c$
$\int \mathrm{d}x = \int 1\, x^0 \, \mathrm{d}x$	$x + c$
$\int \frac{1}{x} \, \mathrm{d}x$	$\ln \lvert x \rvert + c$
$\int \frac{1}{\alpha x + \beta} \, \mathrm{d}x$	$\frac{1}{\alpha} \ln \lvert \alpha x + \beta \rvert + c$
$\int e^x \, \mathrm{d}x$	$e^x + c$
$\int e^{\alpha x + b} \, \mathrm{d}x$	$\frac{1}{\alpha} e^{\alpha x + b} + c \quad \text{if } \alpha \neq 0$
$\int \alpha^x \, \mathrm{d}x$	$\frac{1}{\ln \alpha} \alpha^x + c \quad \text{if } \alpha > 0 \text{ and } \alpha \neq 1$

Indefinite Integral: Properties

- Multiplicative constants

$$\int \alpha f(x)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x$$

- Multiplicative constant -1

$$\int -f(x)\,\mathrm{d}x = -\int f(x)\,\mathrm{d}x$$

- Sums and differences

$$\int [f(x) \pm g(x)]\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$

Indefinite Integral: Examples

Example 1: evaluate the integral

$$\int \left(x^4 - \frac{3}{2x} + e^{2x} - 9\right) \mathrm{d}x = \int x^4 \, \mathrm{d}x - \frac{3}{2}\int \frac{1}{x} \, \mathrm{d}x + \int e^{2x} \, \mathrm{d}x - \int 9 \, \mathrm{d}x$$
$$= \frac{1}{5}x^5 - \frac{3}{2}\ln|x| + \frac{1}{2}e^{2x} - 9x + C$$

Example 2: a common mistake — spot the two errors

$$\int x^4 - \frac{3}{2x}\mathrm{d}x + e^{2x} - 9 = \int x^4 \mathrm{d}x - \frac{3}{2}\int \frac{1}{x}\mathrm{d}x + e^{2x} - 9$$
$$= \frac{1}{5}x^5 - \frac{3}{2}\ln|x| + e^{2x} - 9 + C$$

Definite Integral

- Definite integrals involve integrating a function between two values (the "limits")
- Definition of the definite integral:

$$\int_a^b f(x)\,\mathrm{d}x = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

  where F is any function satisfying $F'(x) = f(x)$ for all $x \in [a, b]$

- (Geometric) interpretation
  - if $f(x) \geq 0$ over $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x$$

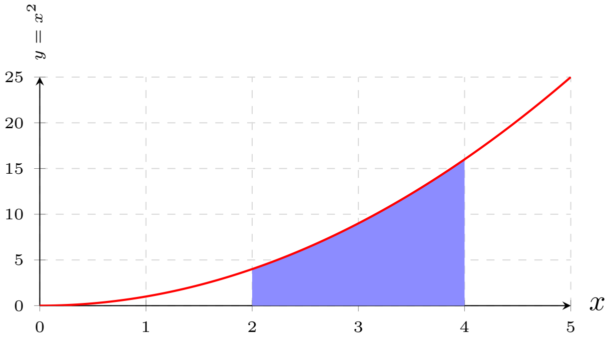
    is the area between the graph of f, the x-axis and the lines $x = a$ and $x = b$.
  - if $f(x) \leq 0$ over $[a, b]$, then

$$-\int_a^b f(x)\,\mathrm{d}x$$

    is the area between the graph of f, the x-axis and the lines $x = a$ and $x = b$.

Definite Integral: Geometric Interpretation

Example: find the area between the x-axis and $f(x) = x^2$ over $[2, 4]$ (shaded area in the figure below)



$$\int_2^4 x^2 \, \mathrm{d}x = \left[\frac{1}{3}x^3\right]_2^4 = \frac{1}{3}4^3 - \frac{1}{3}2^3 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3} \approx 18.66$$

Definite Integral: Properties

- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\int_a^a f(x)\,\mathrm{d}x = 0$
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x, \quad a \leq c \leq b$
- $\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$
- $\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$

Definite Integral: Present Value of a Future Cash Flow

- Net present value: current value of a future reward or payment

$$A = Ve^{-rt}$$

A: net present value $\quad V$: nominal value $\quad r$: discount rate $\quad t$: time to reception

$\rightarrow$ A dollar received later is worth less today, as it could be invested and grow at rate r.

- What about a flow/stream of future payments?
  - Sum of all discounted payments over all periods

$$\Pi = \int_0^{\tau} Ve^{-rt}\,\mathrm{d}t = V\int_0^{\tau} e^{-rt}\,\mathrm{d}t = V\left[-\frac{1}{r}e^{-rt}\right]_0^{\tau} = \frac{V}{r}\left(1 - e^{-r\tau}\right)$$

  - assume $r = 0.06$, $\tau = 5$ and $V = \$3000$
  - present value of the future cash flow: $\Pi = \frac{\$3000}{0.06}\left(1 - e^{-0.06\cdot 5}\right) \approx \$12{,}959$.

$\rightarrow$ Counted naively, five years of payments look like $5 \times \$3000 = \$15{,}000$. The \$2,041 gap is precisely the **cost of waiting**.

Why a *Natural* Exponential?

- **Interest = growth proportional to current value.** In continuous time this is a differential equation, with a unique solution:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = r\,A \qquad \Longrightarrow \qquad A(t) = A_0\, e^{rt}.$$

  Since e^t is the *only* function equal to its own derivative, once growth is proportional the base e is *forced*, not chosen.

- **Why e and no other base?** Only for base e does the exponent's r equal the true instantaneous rate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, a^t = a^t \ln a \quad \Longrightarrow \quad \text{any } a \neq e \text{ smuggles in a stray } \ln a.$$

- **Where e comes from: continuous compounding.**

$$\left(1 + \tfrac{r}{n}\right)^{nt} \xrightarrow[n\to\infty]{} e^{rt}.$$

Discounting is simply this run *backwards* in time: $A = V e^{-rt}$.

Integrals (further information and exercises)

- More comprehensive discussion of integration techniques (on Paul Dawkins' website)
  - Basics and integration by substitution:
    - Explanation:
      `http://tutorial.math.lamar.edu/Classes/CalcI/IntegralsIntro.aspx`
    - Exercises:
      `http://tutorial.math.lamar.edu/Problems/CalcI/IntegralsIntro.aspx`
  - More techniques:
    - Explanation:
      `http://tutorial.math.lamar.edu/Classes/CalcII/IntTechIntro.aspx`
    - Exercises:
      `http://tutorial.math.lamar.edu/Problems/CalcII/IntTechIntro.aspx`
  - Multiple integrals:
    - Explanation:
      `http://tutorial.math.lamar.edu/Classes/CalcIII/MultipleIntegralsIntro.aspx`
    - Exercises:
      `http://tutorial.math.lamar.edu/Problems/CalcIII/MultipleIntegralsIntro.aspx`